

Lattice approach to the $\Delta I = 1/2$ rule¹

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ABSTRACT

We discuss a number of old and new methods for computing $K \rightarrow \pi\pi$ amplitudes on the lattice. They all involve a non-perturbative determination of matching coefficients. We show how problems related to operator mixing can be greatly reduced by using point-split hadronic currents.

1 Introduction

One of the major puzzles remaining in hadronic physics is the so-called “ $\Delta I = 1/2$ rule” in non-leptonic kaon decays. Decays in which isospin changes by $\Delta I = 1/2$ are greatly enhanced over those with $\Delta I = 3/2$. For instance, one finds experimentally

$$\frac{\mathcal{A}(K \rightarrow \pi\pi[I = 0])}{\mathcal{A}(K \rightarrow \pi\pi[I = 2])} \approx 20. \quad (1)$$

Although the origin of this large enhancement is not theoretically well understood, we do know that, in a QCD-based explanation, most of the enhancement must come from long distance, non-perturbative physics.

Let us briefly discuss the source of the difficulties in calculating $\mathcal{A}(K \rightarrow \pi\pi)$ in lattice QCD [1] - [5].

For scales below M_w , but above the charm quark mass, the $\Delta S = 1$ part of the effective weak Hamiltonian can be written as

$$\mathcal{H}_{\text{eff}}^{\Delta S=1} = \lambda_u \frac{G_F}{\sqrt{2}} \left[C_+(\mu, M_w) O^{(+)}(\mu) + C_-(\mu, M_w) O^{(-)}(\mu) \right], \quad (2)$$

$$O^{(\pm)} = \left[(\bar{s} \gamma_\mu^L d)(\bar{u} \gamma_\mu^L u) \pm (\bar{s} \gamma_\mu^L u)(\bar{u} \gamma_\mu^L d) \right] - [u \leftrightarrow c], \quad (3)$$

where $\gamma_\mu^L = \gamma_\mu(1 - \gamma_5)/2$, $\lambda_u = V_{ud}V_{us}^*$, G_F is the Fermi constant and μ is the subtraction point.

The operators $O^{(\pm)}$ have different transformation properties under isospin. In particular, $O^{(-)}$ is pure $I = 1/2$, whereas $O^{(+)}$ contains parts having both $I = 1/2$ and $I = 3/2$. An explanation of the $\Delta I = 1/2$ rule thus requires that the $K \rightarrow \pi\pi$ matrix element of $C_- O^{(-)}$ be substantially enhanced compared to that of $C_+ O^{(+)}$.

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Part of the enhancement is provided by the ratio of Wilson coefficients, C_-/C_+ , in their renormalization group evolution from $\mu \sim M_w$ down to $\mu \sim 2$ GeV, where one finds $|C_-/C_+| \approx 2$. This factor is, however, too small by an order of magnitude to explain the $\Delta I = 1/2$ rule. The remainder of the enhancement must come from the matrix elements of the operators, and these are the quantities that we wish to evaluate on the lattice. Attempts in this direction date back to the works of ref. [1]

There are two major problems which arise in the calculation of $O^{(\pm)}$ matrix elements in lattice QCD.

1. Decay amplitudes into two or more particles cannot be directly accessed in Euclidean space, as a consequence of the Maiani and Testa (MT) no-go theorem [6], except in kinematical configurations where final state interactions are absent.
2. Operators $O^{(\pm)}$ mix with operators of lower dimension with coefficients which diverge as inverse powers of the lattice spacing. These contributions must be computed non-perturbatively and subtracted, leading to prohibitively large statistical errors [5].

In this note we briefly revisit the lattice approach for the case of Wilson fermions (either improved or not)², recalling old methods and reviewing new proposals [8].

2 Old and new approaches

There is a long list of methods proposed in the literature to deal with the problem of computing the hadronic matrix elements that enter in the non-leptonic weak decay amplitude on the lattice. They are all aimed at bypassing the two major difficulties illustrated in the previous section. Here we briefly highlight the merits and the drawbacks of the most promising among them.

- 1) In ref. [3] it was suggested to work with $m_s = m_d$ and calculate the Euclidean amplitude

$$\mathcal{A}_{m_s=m_d}^{(\pm)} = \langle \pi(\vec{p}_1=0) \pi(\vec{p}_2=0) | O^{(\pm)}(\mu) | K(\vec{p}_K=0) \rangle \Big|_{m_s=m_d} \quad (4)$$

with all three particles at rest. Using GIM and CPS-symmetry (CPS= $CP \times s \leftrightarrow d$ interchange [9]), it can be shown that setting $m_s = m_d$ causes all mixings to vanish, removing the need for subtractions. Working with the two pions at rest solves the problem of final state interactions, as they vanish at threshold. The method requires, however, a large extrapolation from the unphysical point $m_s = m_d$ to the physical one with the use of chiral perturbation theory [10].

- 2) An alternative method [8] consists in working with a non-perturbatively $O(a)$ improved fermion action (for which there are no errors of $O(a)$ in the spectrum [11] and on-shell matrix elements of improved currents obey the continuum chiral Ward identities up to $O(a^2)$ [12]). Choosing quark masses such that $m_K = 2m_\pi$, one measures the $K \rightarrow \pi\pi$ amplitudes with all particles at rest. Since with this choice of masses the momentum transfer vanishes ($\Delta p = 0$), one can prove that the matrix element of the dangerous, divergent, subtraction is now of $O(a)$ rather than of $O(1)$ and vanishes in the continuum limit. Furthermore pions are at rest

²For the use of staggered fermions see ref. [7].

and the no-go MT theorem does not apply. Besides the need of a (perhaps less severe) chiral extrapolation, a problem here is the difficulty of tuning with a sufficiently good accuracy the quark masses to have the condition $m_K = 2m_\pi$ satisfied.

3) The old proposal of ref. [2] makes use of the $K \rightarrow \pi$ matrix elements of (the positive parity part of) the weak Hamiltonian and relies on chiral perturbation theory, in the form of Soft Pion Theorems (SPT's), to connect them to $K \rightarrow \pi\pi$ amplitudes. Since only single-particle states are involved, there are no problems with the MT theorem. The disadvantage of the method is that the operator mixing problem is much more severe than for the negative parity part of $\mathcal{H}_{\text{eff}}^W$ (which enters in the $K \rightarrow \pi\pi$ amplitudes), thus making an accurate evaluation of the matrix elements of the renormalized operator very difficult.

The relation between $K \rightarrow \pi\pi$ and $K \rightarrow \pi$ amplitudes is easily found exploiting SPT's. At leading order in chiral perturbation theory the physical amplitude takes, in fact, the form (for $\Delta p = 0$)

$$\langle \pi^+ \pi^- | O^{(\pm)}(\mu) | K^0 \rangle = i \gamma^{(\pm)} \frac{m_K^2 - m_\pi^2}{f_\pi}. \quad (5)$$

As the coefficients $\gamma^{(\pm)}$ appear also in the expression for the $K \rightarrow \pi$ matrix element

$$\langle \pi^+(p) | O^{(\pm)}(\mu) | K^+(q) \rangle = -\delta^{(\pm)} \frac{m_K^2}{f_\pi^2} + \gamma^{(\pm)} p \cdot q. \quad (6)$$

by studying this matrix element on the lattice as a function of $p \cdot q$, one can, in principle, determine $\gamma^{(\pm)}$, from which we obtain the $K \rightarrow \pi\pi$ amplitudes.

In order to construct the finite renormalized lattice operators $O^{(\pm)}$ it was suggested in reference [2] to use perturbation theory to determine finite mixing coefficients and subtract non-perturbatively operators with divergent mixing coefficients. In the case of the parity conserving part of $O^{(\pm)}$ that contribute to eq. (6), this can be elegantly done by adjusting the only divergent coefficient until the momentum independent part of the $K \rightarrow \pi$ matrix element ($\delta^{(\pm)}$ in eq. (6)) vanishes.

4) A method which, in principle, avoids all the difficulties caused by mixing with lower dimension operators and which automatically gives the effective weak Hamiltonian with the correct normalization is based on the idea of studying at short distances ($a \ll |x| \ll \Lambda_{QCD}^{-1}$) the x -behaviour of the OPE expansion of two hadronic currents on the lattice.

The method does not use chiral perturbation theory, and thus applies equally well to the $\Delta S = 1$, $\Delta C = 1$ and $\Delta B = 1$ parts of the weak Hamiltonian. In addition, it allows one to construct an improved weak Hamiltonian (i.e. one having errors of $O(a^2)$), if the improved version of the weak hadronic currents [12] are used.

The method is speculative in the sense that it is likely to require more computational power than is presently available, although we expect it to become practical with the advent of Teraflop machines.

We recall that the standard construction of the non-leptonic weak Hamiltonian begins with the formula

$$\mathcal{H}_{\text{eff}}^W = g_w^2 \int d^4x D_w(x; M_w) T(J_{\rho L}(x) J_{\rho L}^\dagger(0)), \quad (7)$$

where

$$D_w(x; M_w) = \int d^4p \frac{e^{ipx}}{p^2 + M_w^2} \quad (8)$$

is the longitudinal part of the W -boson propagator and $J_{\rho L}$ is the (left-handed) hadronic weak current. One then introduces the operator product expansion (OPE) in the r.h.s of eq. (7), which is justified by the observation that the dominant contribution to the integral comes from distances $|x| \ll M_w^{-1}$. For physical amplitudes, one obtains in this way

$$\langle h | \mathcal{H}_{\text{eff}}^W | h' \rangle = \frac{G_F}{\sqrt{2}} \sum_i C_i(\mu, M_w) M_w^{6-d_i} \langle h | O^{(i)}(\mu) | h' \rangle, \quad (9)$$

where d_i is the dimension of the operator $O^{(i)}(\mu)$, and the functions $C_i(\mu, M_w)$ result from the integration of the Wilson expansion coefficients, $c_i(x; \mu)$ (defined in eq. (11) below), with the W -propagator,

$$C_i(\mu, M_w) M_w^{6-d_i} = \int d^4x D_w(x; M_w) c_i(x; \mu). \quad (10)$$

The $O^{(i)}(\mu)$ are quark and/or gluon operators renormalized at the subtraction point μ . The functions $C_i(\mu, M_w)$ are evaluated in perturbation theory and their running with μ is dictated by the renormalization group equation which follows from the μ -independence of the l.h.s. of eq. (9).

The sum in the expansion (9) is over operators of increasing dimension. As the operator dimension of $\mathcal{H}_{\text{eff}}^W$ is 6, we will have to consider in the following only operators with dimensions $d_i \leq 6$, since the contribution from operators with $d_i > 6$ is suppressed by powers of $1/M_w$.

All the intricacies of operator mixing in the definition of the finite and renormalized operators, $O^{(i)}(\mu)$, come about because the integrals in (7) and (10) are extended down to the region of extremely small x . The complicated mixing of the $O^{(i)}(\mu)$'s in terms of bare operators arises from contact terms when the separation of the two currents goes to zero, i.e. when $|x|$ is of the order of a (the problem is particularly bad here, because chiral symmetry is broken by the lattice regularization). This observation suggests that a simple way to circumvent these difficulties is to directly determine the matrix elements of renormalized operators by enforcing the OPE on the lattice for distances $|x|$ much larger than the lattice spacing a , but much smaller than Λ_{QCD}^{-1} , i.e. in a region where perturbation theory can be used to determine the expected form of the OPE.

We imagine proceeding in the following way. If $J_{\rho L}$ is the appropriately renormalized (and possibly improved) finite lattice current operator, we measure in Monte Carlo simulations the hadronic matrix element $\langle h | T(J_{\rho L}(x) J_{\rho L}^\dagger(0)) | h' \rangle$, as a function of x in the region $a \ll |x| \ll \Lambda_{QCD}^{-1}$. The numbers $\langle h | O^{(i)}(\mu) | h' \rangle$ entering in eq. (9) are extracted by fitting the x -behaviour of $\langle h | T(J_{\rho L}(x) J_{\rho L}^\dagger(0)) | h' \rangle$ to the OPE formula

$$\langle h | T(J_{\rho L}(x) J_{\rho L}^\dagger(0)) | h' \rangle = \sum_i c_i(x; \mu) \langle h | O^{(i)}(\mu) | h' \rangle, \quad (11)$$

where the Wilson coefficients $c_i(x; \mu)$ are determined by continuum perturbation theory using any renormalization scheme we like. The scale μ should be chosen so that $1/\mu$ too lies in the range $a \ll 1/\mu \ll \Lambda_{QCD}^{-1}$. Since we only consider operators of dimension 6 or lower, the lattice T -product differs from the right-hand side of eq. (11) by terms of $O(|x|^2 \Lambda_{QCD}^2)$, which is then an estimate of the size of the systematic errors intrinsic in this procedure. As a last step we insert the numbers $\langle h | O^{(i)}(\mu) | h' \rangle$, determined in this way, in (9), finally obtaining an explicit expression for the matrix elements of $\mathcal{H}_{\text{eff}}^W$.

The procedure illustrated above requires the existence of a window, $a \ll |x| \ll \Lambda_{QCD}^{-1}$, in which the distance between the two currents is sufficiently small that perturbation theory can be used, but large enough that lattice artifacts, which are suppressed by powers of a/x , are tiny. For such a window to exist we need to have an adequately small lattice spacing. At the same time the physical volume of the lattice must be sufficiently large to allow the formation of hadrons.

A few remarks may be useful at this point.

- The method determines directly the “physical” matrix elements of the operators appearing in the OPE of the two currents, i.e. the matrix elements of the finite, renormalized operators $O^{(i)}(\mu)$, without any reference to the magnitude of the W -mass. This means that it will not be necessary to probe distances of $O(1/M_w)$ with lattice calculations.
- Since it is the continuum OPE which determines the operators appearing in the lattice OPE (9), these are restricted by the continuum symmetries. This is because, for $|x| \gg a$, the lattice OPE matches that of the continuum with discretization errors suppressed by powers of a/x .
- The μ -dependence of the matrix elements of the operators $O^{(i)}(\mu)$ is trivially induced by that of the (perturbative) Wilson coefficients, $c_i(x; \mu)$. It compensates the related μ -dependence of the functions $C_i(\mu, M_w)$ in such a way that the l.h.s of eq. (9) is independent of the choice of the subtraction point.
- Unlike the methods discussed before, this approach automatically yields hadronic amplitudes that are properly normalized (in the renormalization scheme in which the Wilson coefficients appearing in eq. (11) are computed).

As for the applicability of this method, we remark that, fortunately, in the case at hand there are no operators of dimension lower than 6. If lower dimension operators were present they would dominate at short distances, since their Wilson coefficients would diverge as powers of $1/x$ (up to logarithmic corrections). In this situation it would be virtually impossible to pick out the matrix elements of the interesting dimension 6 operators.

Operators of dimension 6 have, instead, Wilson coefficients which vary logarithmically with x . At leading order their expression is

$$c_i(x; \mu) \propto \left(\frac{\alpha_s(1/x)}{\alpha_s(\mu)} \right)^{\frac{\gamma_0^{(i)}}{2\beta_0}} = 1 + \frac{\alpha_s}{4\pi} \gamma_0^{(i)} \log(x\mu) + \dots, \quad (12)$$

where $\gamma_0^{(i)}$ is the one-loop anomalous dimension of the operator $O^{(i)}$, and β_0 is coefficient of the one-loop term in the β -function.

In the case of the $\Delta S = 1$ part of $\mathcal{H}_{\text{eff}}^W$, the operators which can appear in (9) are $O^{(\pm)}$ (eq. (3)) and in addition

$$O' = (m_c^2 - m_u^2) \bar{s}(\overrightarrow{D}_\mu - \overleftarrow{D}_\mu) \gamma_\mu^L d. \quad (13)$$

The GIM mechanism causes O' to vanish when $m_c = m_u$, while chiral symmetry requires both the quarks to be left-handed and that the GIM factor be quadratic in the quark masses. Since O' has dimension 6, its coefficient function depends only logarithmically on x .

The anomalous dimensions of the three relevant operators ($O^{(\pm)}$ and O') are

$$\gamma_0^{(+)} = 4, \quad \gamma_0^{(-)} = -8, \quad \gamma'_0 = 16. \quad (14)$$

Actually, the contribution of O' to the r.h.s. of eq. (11) can be determined separately, since its matrix element does not require any subtraction, and needs not be fitted. The anomalous dimensions of the operators $O^{(\pm)}$ are well separated from one another, so it might be possible to determine the amplitudes $\langle h|O^{(\pm)}(\mu)|h' \rangle$ and obtain the physical matrix elements of $\mathcal{H}_{\text{eff}}^{\Delta S=1}$.

3 Conclusions

In this note we have reviewed a number of new approaches aimed at studying the $\Delta I = 1/2$ rule on the lattice, using Wilson-like fermions (similar methods could also be used for staggered fermions).

We have reevaluated the methods of refs. [3] and [2] involving $K \rightarrow \pi\pi$ and $K \rightarrow \pi$ amplitudes, respectively. The last approach is likely to be more difficult than the first one, because of the large number of mixing coefficients which have to be determined non-perturbatively. It may however give complementary information to the results obtained with the $K \rightarrow \pi\pi$ methods and provide a check of the accuracy of chiral relations.

The recently proposed approach [8] based on the study of the short distance behaviour of the OPE of two hadronic weak currents on the lattice is theoretically very appealing. The advent of Teraflop Supercomputers may render it a viable method to directly extract physical amplitudes from Monte Carlo data. An interesting feasibility study in this direction has been undertaken [13]. It consists in the investigation of the small x behaviour of the OPE of two currents in the two-dimensional $O(3)$ σ -model. A preliminary analysis of Monte Carlo data indicates that the measured small x behaviour of the one-particle matrix elements of the product of two currents matches the logarithmic behaviour expected from perturbative calculations, thus allowing the (non-perturbative) evaluation of the matrix elements of the operators appearing in the OPE.

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